EQUIVARIANT CHERN NUMBERS AND THE NUMBER OF FIXED POINTS FOR UNITARY TORUS MANIFOLDS

ZHI LÜ AND QIANGBO TAN

Abstract. Let M^{2n} be a unitary torus $(2n)$-manifold, i.e., a $(2n)$-dimensional oriented stable complex connected closed T^n-manifold having a nonempty fixed set. In this paper we show that M bounds equivariantly if and only if the equivariant Chern numbers $\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M]\rangle = 0$ for all $i, j \in \mathbb{N}$, where $c_l^{T^n}$ denotes the lth equivariant Chern class of M. As a consequence, we also show that if M does not bound equivariantly then the number of fixed points is at least $\lceil \frac{n}{2} \rceil + 1$.

1. Introduction

Let T^n denote the torus of rank n. An *oriented stable complex closed T^n-manifold* is an oriented closed smooth manifold M with an effective T^n-action such that its tangent bundle admits a T^n-equivariant stable complex structure. It is well-known from [GGK] that the equivariant cobordism class of an oriented stable complex closed T^n-manifold with isolated fixed points is completely determined by its equivariant Chern numbers. In this paper, we shall pay more attention on the oriented stable complex (connected) closed T^n-manifolds of dimension $2n$ with nonempty fixed set, which are also called the *unitary torus manifolds* or *unitary toric manifolds* (see [HM] and [M]). These geometrical objects are the topological analogues of compact non-singular toric varieties, and constitute a much wider class than that of quasi-toric manifolds introduced by Davis and Januszkiewicz in [DJ]. Also, the nonempty fixed set of a unitary torus manifold must be isolated since the action is assumed to be effective. In this case, we shall show that the equivariant cobordism class of a unitary torus manifold is determined by only those equivariant Chern numbers produced by the first and the second equivariant Chern classes. Our result is stated as follows.

Theorem 1.1. *Let M be a unitary torus manifold. Then M bounds equivariantly if and only if the equivariant Chern numbers $\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M]\rangle = 0$ for all $i, j \in \mathbb{N}$, where $[M]$ is the fundamental class of M with respect to the given orientation.*

In [K], Kosniowski studied unitary S^1-manifolds and got some interesting results on the fixed points of the action, where "unitary" means that the tangent bundle of M admits an S^1-equivariant stable complex structure. In particular, when the fixed points are isolated, he proposed the following conjecture.

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Conjecture 1.2 (Kosniowski)**.** Suppose that M^n is a unitary S^1-manifold with isolated fixed points. If M does not bound equivariantly then the number of fixed points is greater than $f(n)$, where $f(n)$ is some linear function.

Remark 1. As was noted by Kosniowski in [K], the most likely function is $f(n) = \frac{n}{4}$, so the number of fixed points of M^n is at least $[\frac{n}{4}] + 1$.

With respect to this conjecture, recently some related works have been done. For example, Pelayo and Tolman in [PT] studied compact symplectic manifolds with symplectic circle actions, and proved that if the weights induced from the isotropy representations on the fixed pionts of such a S^1-manifold satisfy some subtle condition, then the action has at least $n+1$ isolated fixed points. In [LL], Ping Li and Kefeng Liu showed that if M^{2mn} is an almost complex manifold and there exists a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of weight m such that the corresponding Chern number $\langle (c_{\lambda_1} \ldots c_{\lambda_r})^n, [M]\rangle$ is nonzero, then for any S^1-action on M, it must have at least $n+1$ isolated fixed points.

In the case of the unitary torus manifolds, comparing with Kosniowski's Conjecture 1.2, we can apply Theorem 1.1 to obtain the following result:

Theorem 1.3. *Suppose that M^{2n} is a $(2n)$-dimenional unitary torus manifold. If M does not bound equivariantly, then the number of fixed points is at least $\lceil \frac{n}{2} \rceil + 1$, where $\lceil \frac{n}{2} \rceil$ denotes the minimal integer no less than $\frac{n}{2}$.*

Remark 2. It should be interesting to discuss whether there exists an example of $(2n)$-dimenional unitary torus manifolds, which doesn't bound euqivariantly but has exactly $\lceil \frac{n}{2} \rceil + 1$ isolated fixed points for every n.

2. Preliminaries

2.1. **Equivariant Chern characteristic numbers.** The *equivariant Chern characteristic numbers* $c_\omega^{T^n}(M)$ of an oriented stable complex closed T^n-manifold M are defined as

$$c_\omega^{T^n}(M) = < (c_1^{T^n})^{i_1} \ldots (c_k^{T^n})^{i_k}, [M] > \in H^*(BT^n; \mathbb{Z})$$

where $\omega = (i_1, \ldots, i_k)$ is a multi-index and $c_l^{T^n}$ is the lth equivariant Chern class of M. Unlike the ordinary Chern characteristic numbers, these equivariant Chern characteristic numbers can be nonzero polynomials in $H^*(BT^n; \mathbb{Z})$ if the degree of the product $\deg(c_1^{T^n})^{i_1} \ldots (c_k^{T^n})^{i_k}$ is greater than $\dim M/2$.

If the oriented stable complex closed T^n-manifold M has only isolated fixed points, then it is known from [GGK] that at each fixed point $p \in M^{T^n}$ the tangent space T_pM is equipped with the induced T^n-action, orientation and complex structure, and the T^n-equivariant cobordism class of M is determined by the complex T^n-representations T_pM at all $p \in M^{T^n}$ and their orientations. Then Guillemin, Ginzburg and Karshon in [GGK] applied Atiyah–Bott–Berline–Vergne localization theorem to give the following theorem.

Theorem 2.1 (Guillemin–Ginzburg–Karshon)**.** *Let M be an oriented stable complex closed T^n-manifold with isolated fixed points. Then M bounds equivariantly if and only if all equivariant Chern characteristic numbers of M are equal to zero.*

2.2. **Unitary torus manifolds and Atiyah–Bott–Berline–Vergne localization theorem.** Let M^{2n} be a $(2n)$-dimensional unitary torus manifold. Following [M], we say that a closed, connected, real codimension-2 submanifold of M^{2n} is called *characteristic* if it is a fixed set component by a certain circle subgroup of T^n and contains at least one T^n-fixed point. Then M^{2n} has finitely many such characteristic submanifolds. By $M_i, i \in [m] = \{1, ..., m\}$ we denote all characteristic submanifolds of M^{2n}, and by ζ_i denote the corresponding normal bundle over M_i, and by T_i denote the circle subgroup fixing M_i pointwise. Then, for each $p \in M^{T^n}$, we can write the tangent T^n-representation at p as

$$T_pM = \bigoplus_{i\in I(p)} \zeta_i|_p$$

where $I(p) = \{i | p \in M_i\} \subset [m]$ and $\zeta_i|_p$ is the restriction of ζ_i to p. So $|I(p)| = n$. Each M_i may define an element λ_i in the equivariant cohomology $H^2_{T^n}(M;\mathbb{Z})$. Actually, the inclusion $M_i \hookrightarrow M$ may induce an equivariant Gysin homomorphism: $H^*_{T^n}(M_i;\mathbb{Z}) \longrightarrow H^{*+2}_{T^n}(M;\mathbb{Z})$, so that $\lambda_i \in H^2_{T^n}(M;\mathbb{Z})$ can be chosen as the image of the identity in $H^0_{T^n}(M_i;\mathbb{Z})$. It was shown in [M, Theorem 3.1] that the total equivariant Chern characteristic class $c^{T^n}(TM)$ of the tangent bundle TM of M can be expressed as

$$c^{T^n}(TM) = \prod_{i\in[m]} (1 + \lambda_i)$$

in $\widehat{H}^*_{T^n}(M;\mathbb{Z}) = H^*_{T^n}(M;\mathbb{Z})/S$-torsion where S is the subset of $H^*(BT^n;\mathbb{Z})$ generated multiplicatively by nonzero elements of $H^2(BT^n;\mathbb{Z})$. It is well-known that the restriction $\lambda_i|_p$ can be regarded as the top equivariant Chern class of $\zeta_i|_p$. Hence, the total equivariant Chern characteristic class of the vector bundle $T_pM \longrightarrow \{p\}$ is

$$c^{T^n}(T_pM) = c^{T^n}(TM)|_p = \prod_{i\in I(p)} (1 + \lambda_i|_p).$$

In particular, Masuda in [M] also showed the following result, which will be very useful in our discussion later.

Lemma 2.2 ([M, Lemma 1.3(1)]). *$\{\lambda_i|_p \big| i \in I(p)\}$ forms a basis of $H^2_{T^n}(\{p\};\mathbb{Z}) \cong H^2(BT^n;\mathbb{Z})$.*

On the other hand, the normal bundle to p in M^{2n} is T_pM with the orientation inherited from M^{2n}. Thus, the equivariant Euler class $e^{T^n}(T_pM)$ of this bundle is $\pm c_n^{T^n}(T_pM) = \pm \prod_{i\in I(p)} \lambda_i|_p$, where the sign is positive if the orientation of T_pM agrees with the complex orientation and negative otherwise.

Each $c^{T^n}(T_pM) = \prod_{i\in I(p)}(1 + \lambda_i|_p) = 1 + \sigma_1(p) + \cdots + \sigma_n(p)$ determines a collection $\sigma(p) = (\sigma_1(p), ..., \sigma_n(p))$, where $\sigma_j(p)$ denotes the jth elementary symmetric function over n variables $\lambda_i|_p, i \in I(p)$. Clearly, $\sigma(p)$ determines the representation T_pM, but not the orientation of T_pM inherited from M.

Now choose a basis $\Lambda = \{\alpha_1, ..., \alpha_n\}$ in $H^2(BT^n;\mathbb{Z})$. Then we obtain a collection $\sigma(\Lambda) = (\sigma_1(\Lambda), ..., \sigma_n(\Lambda))$, where $\sigma_i(\Lambda)$ means the ith elementary symmetric function $\sigma_i(\alpha_1, ..., \alpha_n)$. Set

$$\begin{aligned} m_{\sigma(\Lambda)} &= \sharp\{p \in M^{T^n} \big| \sigma(p) = \sigma(\Lambda), e^{T^n}(T_pM) = \sigma_n(\Lambda)\} \\ &\quad -\sharp\{p \in M^{T^n} \big| \sigma(p) = \sigma(\Lambda), e^{T^n}(T_pM) = -\sigma_n(\Lambda)\} \end{aligned}$$

where we let $m_{\sigma(\Lambda)} = 0$ if $\sigma(\Lambda)$ does not occur as the collection of symmetric functions for any of the T_pM's. With the above understood, now we can state the Atiyah–Bott–Berline–Vergne localization theorem in our case as follows:

Theorem 2.3 (A–B–B–V localization theorem)**.** *Let M^{2n} be a $(2n)$-dimensional unitary torus manifold. Then*

$$c_\omega^{T^n}(M) = \sum_{p\in M^{T^n}} \frac{\sigma_1(p)^{i_1}\dots\sigma_n(p)^{i_n}}{\pm\sigma_n(p)} = \sum_{\sigma(\Lambda)} m_{\sigma(\Lambda)}\sigma_1(\Lambda)^{i_1}\dots\sigma_n(\Lambda)^{i_n-1}$$

where $\omega = (i_1, \dots, i_n)$ is a multi-index.

3. Proofs of main results

First we prove two lemmas which will be used in the proof of Theorem 1.1. Let M^{2n} be a $(2n)$-dimensional unitary torus manifold and let $p, q \in M^{T^n}$ be two fixed points.

Lemma 3.1. *If $\sigma_1(p) = \sigma_1(q)$ and $\sigma_n(p) = \pm\sigma_n(q)$ then $\sigma(p) = \sigma(q)$.*

Proof. If $\sigma_n(p) = \pm\sigma_n(q)$, then $\prod_{i\in I(p)} \lambda_i|_p = \pm\prod_{i\in I(q)} \lambda_i|_q$. So, by Lemma 2.2 we have that $\{\lambda_i|_p\big|i \in I(p)\} = \{\varepsilon_i\lambda_i|_q\big|i \in I(q)\}$ where $\varepsilon_i = \pm1$. Furthermore, if $\sigma_1(p) = \sigma_1(q)$, then

$$\sigma_1(p) = \sum_{i\in I(p)} \lambda_i|_p = \sum_{i\in I(q)} \varepsilon_i\lambda_i|_q = \sum_{i\in I(q)} \lambda_i|_q = \sigma_1(q)$$

so $\sum_{i\in I(q)}(1-\varepsilon_i)\lambda_i|_q = 0$. This implies that $\varepsilon_i = 1$ for all $i \in I(q)$ since $\lambda_i|_q, i \in I(q)$ are linearly independent, and the lemma then follows. $\square$

Lemma 3.2. *$\sigma(p) = \sigma(q)$ if and only if $\sigma_1(p) = \sigma_1(q)$ and $\sigma_2(p) = \sigma_2(q)$.*

Proof. It suffices to show that $\sigma(p) = \sigma(q)$ if $\sigma_1(p) = \sigma_1(q)$ and $\sigma_2(p) = \sigma_2(q)$. Consider $s_2(p) = \sum_{i\in I(p)}(\lambda_i|_p)^2$ and $s_2(q) = \sum_{i\in I(q)}(\lambda_i|_q)^2$. If $\sigma_1(p) = \sigma_1(q)$ and $\sigma_2(p) = \sigma_2(q)$, then $s_2(p) = s_2(q)$ since $s_2 = \sigma_1^2 - 2\sigma_2$ by [MS]. Since both $\{\lambda_i|_p\big|i \in I(p)\}$ and $\{\lambda_i|_q\big|i \in I(q)\}$ are two bases of $H^2(BT^n;\mathbb{Z})$ by Lemma 2.2, there is an $n\times n$ non-degenerate $\mathbb{Z}$-matrix A such that

$$(\lambda_i|_p\big|i \in I(p)) = (\lambda_i|_q\big|i \in I(q))A.$$

Moreover, we have that

$$s_2(p) - s_2(q) = (\lambda_i|_q\big|i \in I(q))(AA^\top - E_n)(\lambda_i|_q\big|i \in I(q))^\top = 0$$

so we conclude that $AA^\top = E_n$, where E_n is the identity matrix. This implies that each row of A contains only one ±1 and the other elements in this row are all 0. Hence

$$\sigma_n(p) = \pm\prod_{i\in I(q)} \lambda_i|_q = \pm\sigma_n(q)$$

and then the proof is completed by Lemma 3.1. $\square$

Let $\{\sigma_1(p)\big|p \in M^{T^n}\} = \{\tau_1, \dots, \tau_s\}$ and $\{\sigma_2(p)\big|p \in M^{T^n}\} = \{\eta_1, \dots, \eta_u\}$. Set

$$\mathcal{A}_k = \{p \in M^{T^n}\big|\sigma_1(p) = \tau_k\}$$

for $1 \le k \le s$ and

$$\mathcal{B}_l = \{p \in M^{T^n}\big|\sigma_2(p) = \eta_l\}$$

for $1 \le l \le u$. Then $|M^{T^n}| = \sum_{k=1}^{s} |\mathcal{A}_k| = \sum_{l=1}^{u} |\mathcal{B}_l|$.

Proof of Theorem 1.1. By Theorem 2.1 it suffices to prove that if the equivariant Chern numbers $\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M]\rangle = 0$ for all $i, j \in \mathbb{N}$, then M^{2n} bounds equivariantly. Now suppose $\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M]\rangle = 0$ for all $i, j \in \mathbb{N}$. By Theorem 2.3, we can write these equivariant Chern numbers in the following way:

$$\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M]\rangle = \sum_{k=1}^{s} \tau_k^i \sum_{l \in \mathcal{L}_k} \eta_l^j \sum_{\substack{\sigma_1 = \tau_k \\ \sigma_2 = \eta_l}} \frac{m_\sigma}{\sigma_n} \tag{3.1}$$

where $\mathcal{L}_k = \{l \big| \mathcal{A}_k \cap \mathcal{B}_l \neq \emptyset, 1 \le l \le u\}$. Obviously, $|\mathcal{L}_k| \le |\mathcal{A}_k|$ for every k. Let i vary in the range $0, 1, \ldots, s-1$. Then (τ_k^i) is an $s \times s$ van der Monde matrix, so for each k,

$$\sum_{l \in \mathcal{L}_k} \eta_l^j \sum_{\substack{\sigma_1 = \tau_k \\ \sigma_2 = \eta_l}} \frac{m_\sigma}{\sigma_n} = 0.$$

Next, let j vary in the range $0, 1, \ldots, |\mathcal{L}_k| - 1$. Then (η_l^j) is a $|\mathcal{L}_k| \times |\mathcal{L}_k|$ van der Monde matrix, hence

$$\sum_{\substack{\sigma_1 = \tau_k \\ \sigma_2 = \eta_l}} \frac{m_\sigma}{\sigma_n} = 0.$$

Furthermore, by Lemma 3.2 we have that $\frac{m_\sigma}{\sigma_n} = \sum_{\substack{\sigma_1 = \tau_k \\ \sigma_2 = \eta_l}} \frac{m_\sigma}{\sigma_n} = 0$ so $m_\sigma = 0$ for all σ. Thus, by Theorem 2.3, all equivariant Chern characteristic numbers of M are equal to zero, as desired. □

Now we focus on the proof of Theorem 1.3. First we give a general result.

Proposition 3.3. *If* $s + 2 \max\limits_{1 \le k \le s} \{|\mathcal{A}_k|\} - 3 < n$ *or* $2u + \max\limits_{1 \le l \le u} \{|\mathcal{B}_l|\} - 3 < n$, *then* M *bounds equivariantly.*

Proof. In a similar way to the proof of Theorem 1.1, we can write the equivariant Chern numbers $\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M]\rangle$ in the following two ways:

$$\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M]\rangle = \sum_{k=1}^{s} \tau_k^i \sum_{l \in \mathcal{L}_k} \eta_l^j \sum_{\substack{\sigma_1 = \tau_k \\ \sigma_2 = \eta_l}} \frac{m_\sigma}{\sigma_n} \tag{3.2}$$

and

$$\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M]\rangle = \sum_{l=1}^{u} \eta_l^j \sum_{k \in \mathcal{K}_l} \tau_k^i \sum_{\substack{\sigma_1 = \tau_k \\ \sigma_2 = \eta_l}} \frac{m_\sigma}{\sigma_n} \tag{3.3}$$

where $\mathcal{L}_k = \{l \big| \mathcal{A}_k \cap \mathcal{B}_l \neq \emptyset, 1 \le l \le u\}$ as before and $\mathcal{K}_l = \{k \big| \mathcal{A}_k \cap \mathcal{B}_l \neq \emptyset, 1 \le k \le s\}$, satisfying that $|\mathcal{L}_k| \le |\mathcal{A}_k|$ for every k and $|\mathcal{K}_l| \le |\mathcal{B}_l|$ for every l. We note that if $i + 2j < n$, then $\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M]\rangle = 0$. If $s + 2 \max\limits_{1 \le k \le s} \{|\mathcal{A}_k|\} - 3 < n$, then we can let i vary in the range $0, 1, \ldots, s-1$ and for every k, let j vary in the range $0, 1, \ldots, |\mathcal{L}_k| - 1 \le \max\limits_{1 \le k \le s} \{|\mathcal{A}_k|\} - 1$ in Equation (3.2). Similarly, if $2u + \max\limits_{1 \le l \le u} \{|\mathcal{B}_l|\} - 3 < n$, then we can let j vary in the range $0, 1, \ldots, u-1$ and for every l, let i vary in the range $0, 1, \ldots, |\mathcal{K}_l| - 1 \le \max\limits_{1 \le l \le u} \{|\mathcal{B}_l|\} - 1$ in Equation (3.3). Using the proof method of Theorem 1.1 as above, we can get van der Monde

matrices, which imply that $m_\sigma = 0$ for all σ, and hence $\langle (c_1^{T^n})^i (c_2^{T^n})^j, [M]\rangle = 0$ for all $i, j \in \mathbb{N}$. Therefore, M bounds equivariantly by Theorem 1.1. □

Lemma 3.4. *Let $a_1, \dots, a_r$ be positive integers. If $a_1 + \cdots + a_r = \ell$, then $r + 2\max\{a_i | 1 \le i \le r\} \le 2\ell + 1$.*

Proof. Obviously, $\max\{a_i | 1 \le i \le r\} \le \ell - r + 1$, and the equation holds if and only if there is only some one $a_i = \ell - r + 1$ and all others are equal to 1. Then we have the required inequality $r + 2\max\{a_i | 1 \le i \le r\} \le 2\ell + 1$, where the last equation holds if and only if $r = 1$. □

Proof of Theorem 1.3. If $|M^{T^n}| = |\mathcal{A}_1| + \cdots + |\mathcal{A}_s| < \frac{n}{2} + 1$, then by Lemma 3.4, $s + 2\max\limits_{1\le k\le s}\{|\mathcal{A}_k|\} \le 2|M^{T^n}| + 1 < n + 3$, so M bounds equivariantly by Proposition 3.3. □

Remark 3. Let us look at the case in which M doesn't bound equivariantly and $|M^{T^n}| = \lceil \frac{n}{2} \rceil + 1$. When n is even, we have $s + 2\max\limits_{1\le k\le s}\{|\mathcal{A}_k|\} = 2|M^{T^n}| + 1$ by Proposition 3.3 and Lemma 3.4. This implies that $s = 1$ by the proof of Lemma 3.4, which means that all σ_1 are the same. When n is odd, we see that $n + 3 \le s + 2\max\limits_{1\le k\le s}\{|\mathcal{A}_k|\} \le 2|M^{T^n}| + 1 = n + 4$. An easy argument shows that $n + 3 = s + 2\max\limits_{1\le k\le s}\{|\mathcal{A}_k|\}$ is impossible, so we must have $s + 2\max\limits_{1\le k\le s}\{|\mathcal{A}_k|\} = 2|M^{T^n}| + 1$. Thus, in this case s must be 1 and then all σ_1 are the same, too. Moreover, in a similar way to the proof of Theorem 1.1, we can show easily that $|M^{T^n}| = u$ so all σ_2 are distinct. These observations seemingly imply the existence of a nonbounding unitary torus manifold M^{2n} with $|M^{T^n}| = \lceil \frac{n}{2} \rceil + 1$. Indeed, we can see an example in the case $n = 1$, as shown in [K, Theorem 5].

Finally we conclude this paper with the following conjecture:

Conjecture 3.5. $\lceil \frac{n}{2} \rceil + 1$ is the best possible lower bound of the number of fixed points for $(2n)$-dimensional nonbounding unitary torus manifolds.

References

[DJ] M. W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus Actions*, Duke Mahtematicial Journal, **62** (1991), 417–451.

[GGK] V. Guillemin, V. Ginzburg and Y. Karshon, *Moment Maps, Cobordisms, and Hamiltonian Group Actions*, Mathematical Surveys and Monographs, **98**, American Mathematical Society, Providence, RI, 2002.

[HM] A. Hattori and M. Masuda, *Theory of multi-fans*, Osaka J. Math. **40** (2003), 1–68.

[K] Czes Kosniowski, *Some formulae and conjectures associated with circle actions*, Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979), pp331–339, Lecture Notes in Math., **788**, Springer, Berlin, 1980.

[LL] Ping Li and Kefeng Liu, *Some remarks on circle action on manifolds*, arXiv: 1008.4820, to appear in Mathematical Research Letters.

[M] M. Masuda, *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J. **51** (1999), 237–265.

[MS] J.W. Milnor and J.D. Stasheff, *Characteristic Classes*, Ann. of Math. Stud., vol. **76**, Princeton University Press/University of Tokyo Press, Princeton, NJ/Tokyo, 1974.

[PT] A. Pelayo and S. Tolman, *Fixed points of symplectic periodic flows*, arXiv: 1003.4787, to appear in Ergodic Theory and Dynamical Systems.

School of Mathematical Sciences and The Key Laboratory of Mathematics for Nonlinear Sciences of Ministry of Education, Fudan University, Shanghai, 200433, P.R. China.

E-mail address: zlu@fudan.edu.cn

School of Mathematical Sciences, Fudan University, Shanghai, 200433, P.R. China.

E-mail address: 081018011@fudan.edu.cn